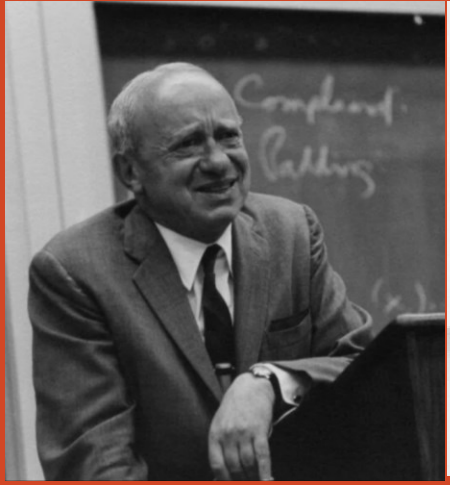


Historical Debates among Constructivists

Ioannis Vandoulakis

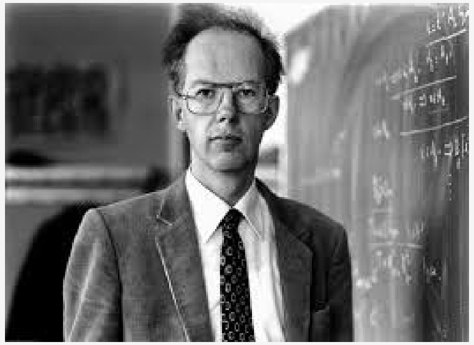
Open University of Cyprus, Cyprus

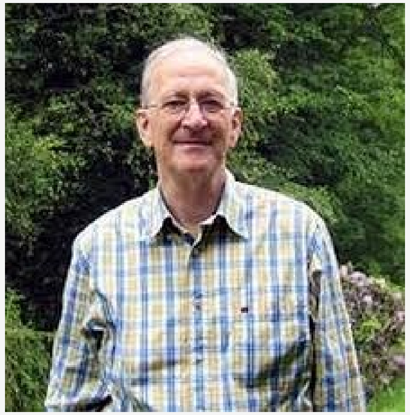
"Semiotics of Logic and Reasoning" Research Group

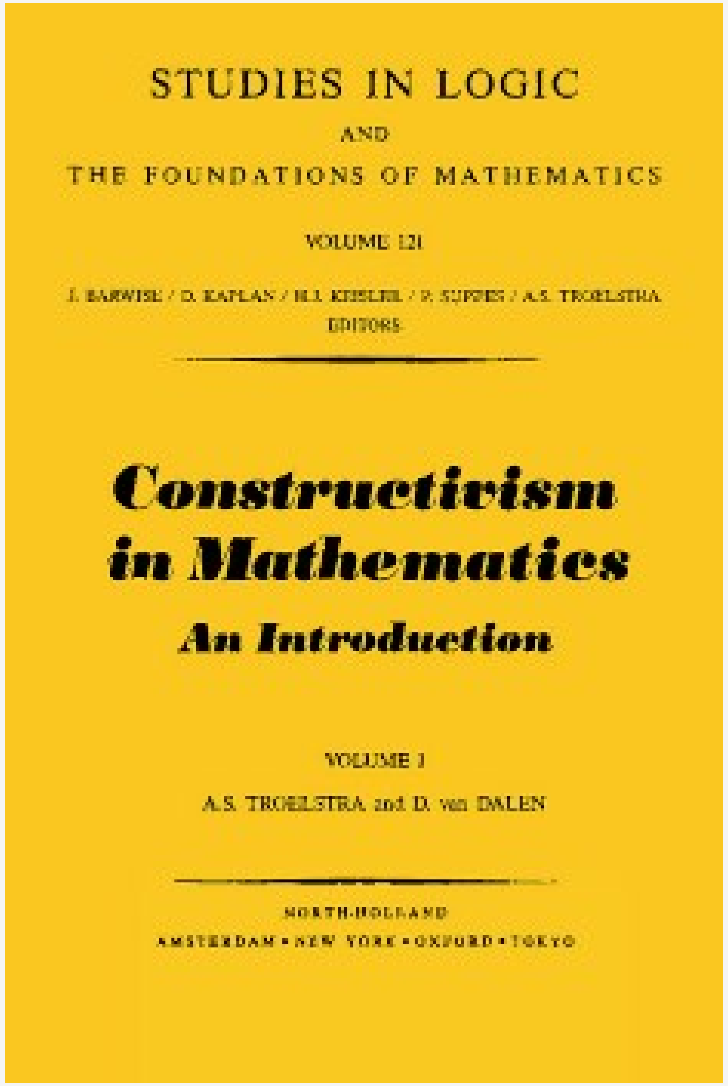
Historical Varieties of Constructivism

In their book, *Constructivism in Mathematics. An Introduction* (Vol 1) Anne Troelstra (1939-2019) and Dirk van Dalen focus on *constructivism*, understood in the "wide sense," which covers in particular

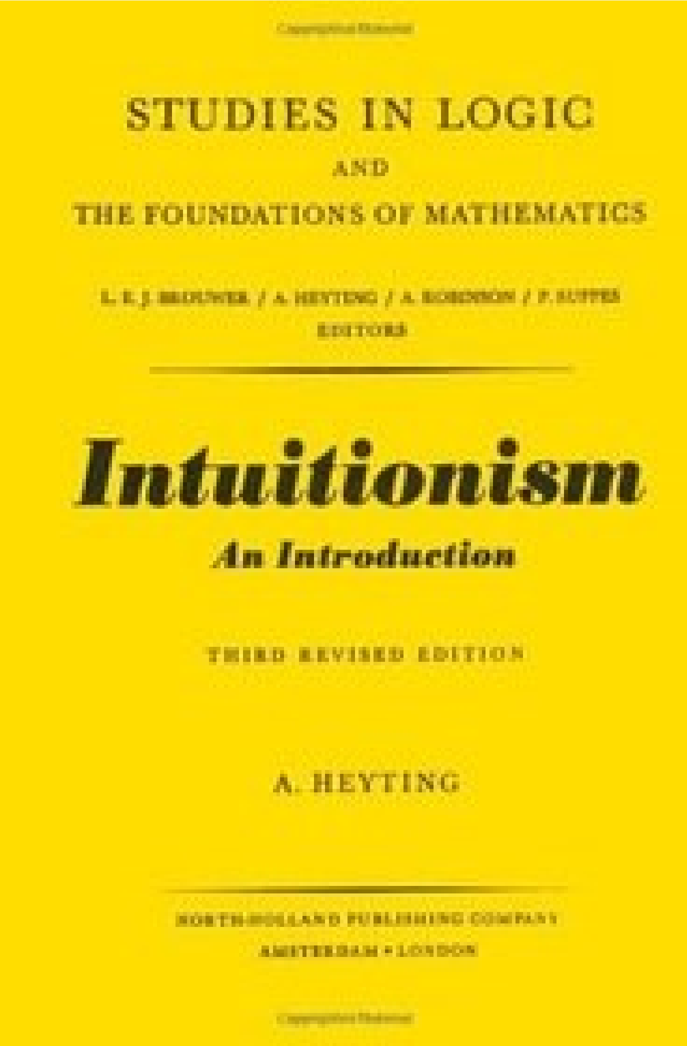
- Brouwer's intuitionism,
- Bishop's constructivism, and
- A.A. Markov's constructive recursive mathematics."





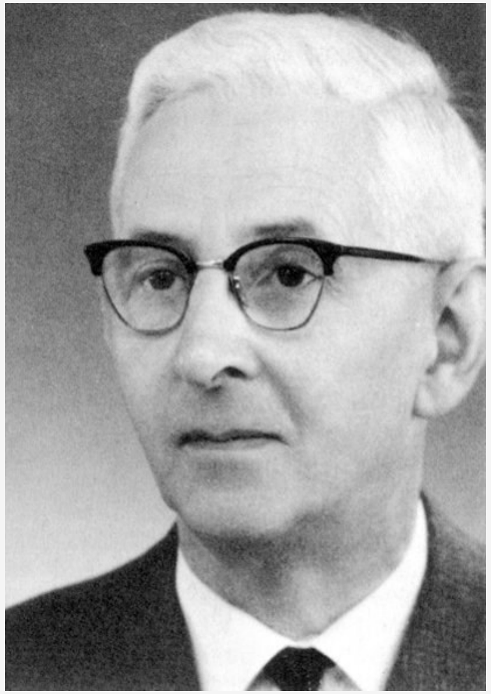


Historical Varieties of Constructivism



Moreover, they note that

> "The first successful introduction to constructive mathematics, more specifically, to intuitionistic mathematics, was Heyting's *Intuitionism, an introduction*, which first appeared in 1956 and went through three editions."



Historical Varieties of Constructivism

In our presentation, we'll see that Heyting's Intuitionism played a special role in establishing intellectual contact between Brouwer's intuitionism and Markov's constructivism.

In addition to the three varieties of constructivism above, Troelstra and van Dalen provide a brief overview of the various historical forms of constructivism, with rather perplexing relations among them.

They list the following forms of constructivism:

1. Finitism.

- Kronecker (1823-1891),
- Thoralf Skolem (1887–1963), in his paper "The foundations of elementary arithmetic" (1923),
- Reuben Goodstein (1912–1985), in his books *Recursive Number Theory* (1957), *Recursive Analysis* (1961)
- Hilbert's program (Hilbert's finitism).

Historical Varieties of Constructivism

2. Predicativism.

- Hermann Weyl (1885-1955) *Das Kontinuum* (1918).
- Paul Lorenzen (1915–1994), in his books *Einfishrung in die operative Logik und Mathematik* (1955) and *Differential und Integral* (1965).

3. The French semi-intuitionists.

- René-Louis Baire (1874–1932),
- Emile Borel (1871-1956),
- Henri Lebesgue (1875–1941),
- Henri Poincaré (1854-1913)

Historical Varieties of Constructivism

4. The Moscow semi-intuitionists

- Nikolai N. Luzin (1883–1950)
  [included in the previous group by Troelstra and van Dalen].

5. Kolmogorov's interpretation of intuitionism as the *calculus of problems* (Brouwer-Heyting-Kolmogorov interpretation)

- Andrey N. Kolmogorov (1903–1987)

6. Esenin-Vol'pin's ultra-finitism.

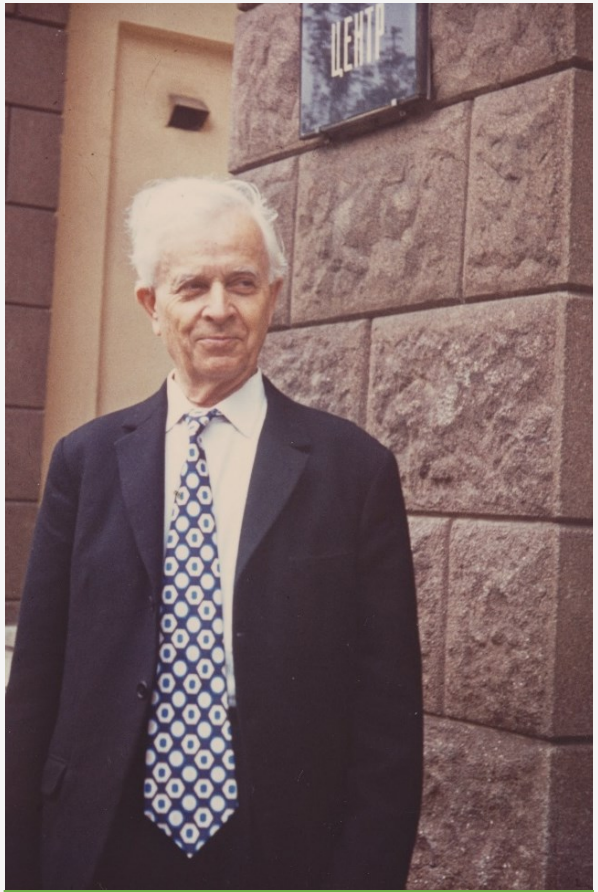
- Petr K. Rashevskii (1907–1983)
  [not mentioned by Troelstra and van Dalen].
- Alexander S. Esenin-Volpin's "ultra-intuitionism" (1924–2016).

1. Markov vs. Brouwer

Although we have several works by Brouwer on his philosophy of intuitionism, Andrei A. Markov (1903-1979) did not write any papers concerning the philosophy underlying his constructive mathematics.

Nikolai M. Nagorny (1928–2007) supposed that Markov was a positivist, but Boris A. Kushner (1941–2019) believed that Markov had a philosophical standpoint. He was led to this conclusion by witnessing an encounter between Markov and Errett A. Bishop (1928–1983) in 1966 during the International Mathematical Congress held in Moscow.

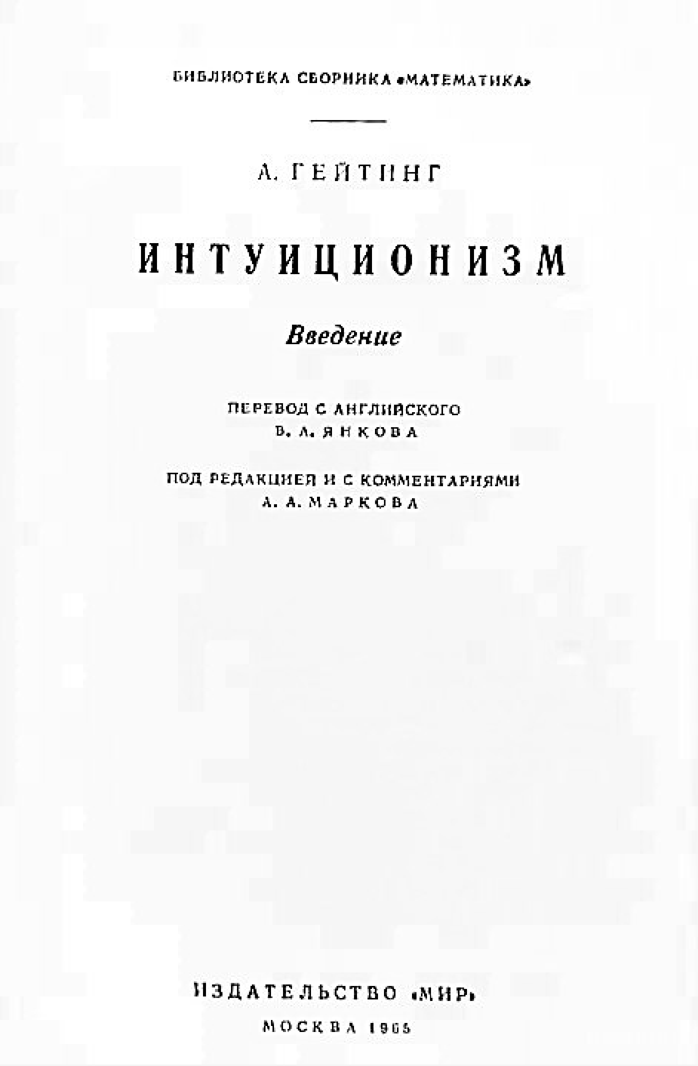
However, what was Markov's 'constructivist' philosophy about?



A.A. Markov. Photo donated by N.M. Nagorny.

1. Markov vs. Brouwer

We would not have been able to get an idea of Markov's 'constructivist' methodological conception, if Yankov had not the brilliant idea, while he translated Heyting's *Intuitionism* into Russian, to suggest to Markov (who was the editor) to respond to Heyting's fictional persons "Class," "Form," "Int," "Pragm," and "Sign," which represent the classical mathematician, formalist, intuitionist, pragmatist and significist, respectively.


БИБЛИОТЕКА СБОРНИКА «МАТЕМАТИКА»

А. ГЕЙТИНГ

ИНТУИЦИОНИЗМ

Введение

ПЕРЕВОД С АНГЛИЙСКОГО
В. А. ЯНКОВА

ПОД РЕДАКЦИЕЙ И С КОММЕНТАРИЯМИ
А. А. МАРКОВА

ИЗДАТЕЛЬСТВО «МИР»
МОСКВА 1965

1. Markov vs. Brouwer

Thus, “Con” enters a peculiar one-sided dialogue with the team of the other representatives, which he can criticize, without himself being liable to criticism!

This response is the only historical record of Markov’s views on Brouwer’s intuitionistic mathematics and logic.

“Con” focuses his criticism predominantly against Brouwer, whom he perceives as his principal antagonist, and essentially ignores the other representatives, except David Hilbert, to whom he devotes an ironic comment about his program to “save” the “precious” mathematical results that lacked content (“what to save and why?”). Markov rejected classical mathematics, specifically mathematical theorems about the abstract existence of mathematical objects, obtained by indirect proof that cannot be found by an ‘algorithmic’ procedure.

1. Markov vs. Brouwer

Based on "Con"'s comments and criticism, we can reconstruct the differences between Markov's constructive mathematics and logic and Brouwer's intuitionistic mathematics and logic as viewed by Markov.

These differences concern:

1. **Different understanding of mathematical objects**:
   Mentally constructed objects in intuitionism; real objects obtained as an outcome of a process executable in a computer, in Markov's constructivism.

1. Markov vs. Brouwer

In Markov's view, constructive mathematics studies *constructive processes* and the *constructive objects* they generate. However,

- The *process of construction* is not extra-linguistic or mental, as in Brouwer, but real, as a process executable on a computer.

- By *constructive objects*, are understood not mentally conceived objects, but concrete objects, like the letters of an alphabet, that is, a (finite or infinite) collection of discernible signs.

1. Markov vs. Brouwer

Markov rejects Brouwer's assumption – that is grounded on Kant's concept of *intuition* (*Anschauung*) – that certain objects of mathematics and mathematical operations are sufficiently evident, so that the manipulation of these objects by such operations cannot lead to inconsistencies and bypasses Brouwer's view of mathematical propositions (formulae, equations, etc.) as mere *images* (*Bilder*) of free mathematical creation.

1. Markov vs. Brouwer

2. **The idea of the infinite**.
   Markov introduces the *abstraction of potential realizability*, which abstracts from any practical spatial, temporal, or material limitations on our capacity to construct (concrete or abstract) mathematical objects.

Markov understands Brouwer's mental constructions as potentially realizable, since they have (practically) realizable material constructions as archetypes. In this way, he reinterprets Brouwer's idea of *potential infinity* in terms of his *abstraction of potential realizability*, in an attempt to "return it down to earth".

1. Markov vs. Brouwer

3. **Mathematical existence.**
   Markov identifies mathematical existence not with **constructability**, as Brouwer does, but with the potential realizability of a construction. However, this construction is not perceived as a process evolving over time, like in Brouwer's concept of *creative subject* or *Kripke's scheme*.

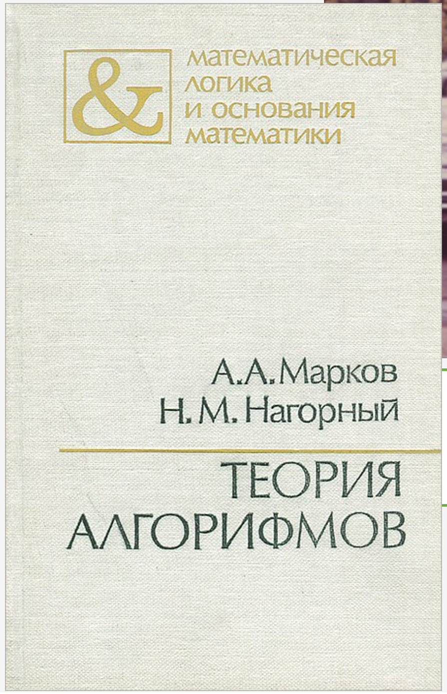
An object exists whenever it can be indicated as a complete finite word (in an alphabet), or it is given by a pair (letter, algorithm), and it is known that the algorithm is applicable to the letter. If such a pair cannot be constructed or the applicability of the algorithm cannot be established, this does not mean that the object does not exist.

An object does not exist only whenever the impossibility of the object's being constructed is proved (for instance, the inapplicability of the corresponding algorithm). In this case, the object under consideration does not exist eternally.

1. Markov vs. Brouwer

4. **Normal algorithms**.
   Markov rejects Brouwer's concept of *infinitely proceeding sequences* and substitutes it with his concept of the *normal algorithm* that he discovered in his study on the identity problem for semigroups.

Markov and Russian constructivists use the term алгорифм, instead of the common Russian term алгоритм [Марков, 1954]



A.A. Markov and N.M. Nagorny.
Photo donated by N.M. Nagorny.

1. Markov vs. Brouwer

Markov considered that Brouwer's concept of *choice sequence* is not evident and, possibly, non-constructive.

It is not sufficiently clear why Markov considers that Brouwer's concept of choice sequence is not evident. A possible explanation is the impossible practical implementation of the acts of choice due to their random character. This explanation is suggested by the following comment of Markov:

1. Markov vs. Brouwer

> It grieves me to see somebody whom you are ready to coerce, so that he would make so many "acts of choice" or "dice drops." My understanding of choice sequences is more human, since algorithms can be executed easily by a computer. And what is most important is that my understanding is constructive. Because the concept of an algorithm can be standardized, it makes it possible to code an algorithm and record it by "letters" in a fixed alphabet. In turn, algorithms themselves can become constructive objects. It is possible to apply other algorithms to them, which is very important in constructive analysis.
>
> Your choice sequences are not constructive objects, and I cannot deal with them. [Heyting (Markov) 1956, 166].

In Nagorny's opinion, Markov's rejection of the choice sequences as non-constructive was a misunderstanding.

1. Markov vs. Brouwer

Accordingly, Markov reinterprets Heyting's concept of *spread* (given in terms of the *spread-law* and the *complementary law*) by using the concept of a *normal algorithm* instead of the concept of *law*.

However, under the new understanding of the concept of spread (in Markov's sense), Brouwer's *Fan Theorem*

> [If there exists a rule with the aid of which a certain object — say, a natural number — can be assigned to each element of a fan, then there exists a natural number *z* such that for each element of the fan this object is defined by the first *z* values of the element.]

is no longer true.

1. Markov vs. Brouwer

5. **The Church Thesis and the Principle of Normalization of Algorithms**
   In Markov's constructive mathematics, the *Church Thesis* assumes the form of the *principle of normalization of algorithms*, which states that every verbal algorithm in an alphabet *V* is equivalent with respect to *V* to some *normal algorithm* in *V*, or, concisely, every verbal algorithm is normalizable.

Markov thinks of his principle as a version of the Church Thesis (Alonso Church, 1903-1995), which expresses the fact that certain refinements of the concept of algorithm (such as, for instance, the concepts of *recursive function*, of λ-*definable function*, etc.) are adequate explications of the general concept of algorithm. Thus, the *principle of normalization of algorithms* expresses the analogous fact with respect to normal algorithms.

1. Markov vs. Brouwer

In connection with Brouwer’s examples depending on unsolved problems, Markov develops a lengthy critical argument in defense of the Church Thesis, which is not acceptable by Brouwer and the intuitionists.

> “Your examples are very pleasant and subtle. Each of them is based on some problem, unsolved at present. You are obviously convinced that as soon as the problem you use is solved in one or another way (which may comfortably happen), you will immediately invent another example of the same kind, based on another unsolved problem.

1. Markov vs. Brouwer

Let us free ourselves to imagine that some genius mathematician invented a general method (an "algorithm") that enables us to solve any single mathematical problem, that is, to give a correct answer, "yes" or "no", to any mathematical question requiring such an answer. Then you will not be able to invent any such problem, and you would be apparently compelled to agree on everything with Mr. Class [the representative of classical mathematics]. You are possibly afraid of such a tragic perspective. Of course, you are aware that Church proved the undecidability of the decidable problem [Church 1936] and that several modest "massive" mathematical problems have been proved today. However, all these results are based on one or the other refinement of the concept of algorithm ("unified general method"), for instance, on the concept of recursive function, and the assumption about the adequacy of this refinement, for instance, on Church Thesis asserting that "recursiveness" is equivalent to "calculability."

1. Markov vs. Brouwer

It is clear that without digging into the concept of algorithm, no proof of impossibility of decidable algorithm may pass through. If you were not willing to accept Church Thesis or some version of it, then you would be compelled to agree that your divergence with Mr. Class depends on the state of our knowledge at the present time. All your anti-classical propositions should be then considered as de facto truths, not as de jure truths. Are you comfortable with that?

On the other hand, if you accept Church Thesis and the modern theory of algorithms, then this would entail a substantial reform of your mathematical outlook and the transition from intuitionism to constructive understanding of mathematics." [Heyting (Markov) 1956, 163-64]

1. Markov vs. Brouwer

6. The concept of number and the continuum

Markov's concept of natural numbers is essentially the same as the intuitionistic understanding as presented in Heyting's *Intuitionism*.
Rational numbers are understood as words of a certain type over the alphabet $\{|, -, /\}$, where "−" is the sign of *minus*, "/" is the sign of *fraction*.
A *(constructive) sequence of rational numbers* is a (normal) algorithm that maps every natural number into a rational number.
A pair of normal algorithms (encoded appropriately by a word) is a constructive real number if the first algorithm is a constructive sequence of rational numbers and the second effectively estimates the rate of convergence of this sequence.

Markov's continuum has properties that do not occur in the classical continuum. For instance, all constructive real functions are continuous, i.e., no real function can have a constructive discontinuity at any point.

1. Markov vs. Brouwer

7. Not all intuitionistic theorems are true in constructive mathematics and vice versa

Markov also attacks certain intuitionistic theorems that are not true in his constructive mathematics, outlining a rather complicated picture:

- Some intuitionistic theorems are refutable in Markov's constructive mathematics (e.g., Heine-Borel theorem [Заславский 1962]), whereas

- there are theorems in Markov's constructive mathematics that do not hold in intuitionistic mathematics (e.g., that every constructive function of a real variable is continuous everywhere in its domain of definition [Марков 1958]).

1. Markov vs. Brouwer

However, certain radical divergences of Markov's viewpoint from Brouwer's intuitionism are not considered in Markov's critical comments to Brouwer's intuitionism, for instance, the adoption of the so-called

8. **Markov's Principle or the Principle of Constructive Selection** [Марков, 1962].

In its general form, it asserts that

> *if a constructive process, given by some prescription, is not potentially infinite (unboundedly extendable), then it terminates.*

Or, concisely,

> *If it is impossible that an algorithm never halts, then it halts.*

1. Markov vs. Brouwer

The intuitive justification of the constructive selection principle within the framework of the system of abstractions applicable in constructive mathematics consists in the following:

> If the impossibility of the potential infinity of a given constructive process is conclusively proved, then the termination of this process is potentially attainable as a result of carrying it out sequentially, step by step.
> Thus, in the assertion of the existence of a constructive object (for example, the result of applying a normal algorithm to a word) on the basis of the constructive selection principle, one is not going beyond the framework of the abstraction of potential realizability.

1. Markov vs. Brouwer

Let A be an algorithmically-verifiable property of natural numbers. If the proposition that there does not exist a number with property A is refuted, then there is a natural number with property A. The corresponding logical scheme is written in the form

$$\forall x(A(x) \vee \neg A(x)) \supset (\neg\neg\exists x A(x) \supset \exists y A(y)).$$

The constructive selection principle is absolutely admissible from the point of view of classical logic, since it is a special case of the general rule of removing a double negation and the Law of the Excluded Middle (LEM). Hence, this semi-classical principle is not admissible by the intuitionists and remained controversial and insufficiently evident even among the Markovian constructivists [Kushner, 1973, 45].

1. Markov vs. Brouwer

9. No comment on Brouwer's intuitionistic logic.

It is curious that Markov did not make any comment on Brouwer's intuitionistic logic. Neither does he contrast or compares his constructive logic with Brouwer's intuitionistic logic.

The semantics for Markov's constructive logic has deep differences from that of intuitionistic logic and was a later development; it is based on the idea of a *hierarchy of (formal) languages* [Markov 1974a-g]

1. Markov vs. Brouwer

10. Constructive Mathematics is a technological science.

Markov did not fully develop this idea in his comments to Heyting's *Intuitionism*.

He criticizes Heyting's view that mathematics studies certain functions of the human mind and therefore it is more akin to philosophy, history, and the social sciences [Heyting 1971, 10].

Markov's objection is based on the argument that a human, together with his mental constructions, is part of nature. Mental constructions, such as the construction of greater and greater natural numbers, have material archetypes in reality. Moreover, mental constructions, such as complex algorithms, are initially conceived as mental constructions but are implemented afterwards as computer programs. Consequently, mental constructions are not considered by Markov as falling under social sciences.

1. Markov vs. Brouwer

This line of argumentation is advanced in an unfinished manuscript written during the last months of Markov's life, published by N. Nagorny in 1987. In this manuscript, it is stated that

> "Since [constructive mathematics] investigates and supplies instruments, applied in various spheres of human activity, in this sense, it is like engineering" [Markov 1987, 212].

Thus, Markov's view on constructive mathematics is a viewpoint of a specialist primarily interested in the applications of mathematics. His viewpoint can be characterized more precisely as a metamathematical (methodological) one with naturalistic philosophical underpinnings.

1. Markov vs. Brouwer

11. Heyting's response

In the third edition of his *Introduction*, in 1971, Heyting states that

> "Many smaller corrections were suggested by the notes which A. A. Markov added to the Russian translation of the first edition"

without, however, distinguishing or highlighting these corrections.

2. Markov vs. Bishop

Boris Abramovich Kushner (1941–2019) witnessed a curious encounter between Markov and Errett Albert Bishop (1928–1983) in 1966 at the International Mathematical Congress in Moscow.

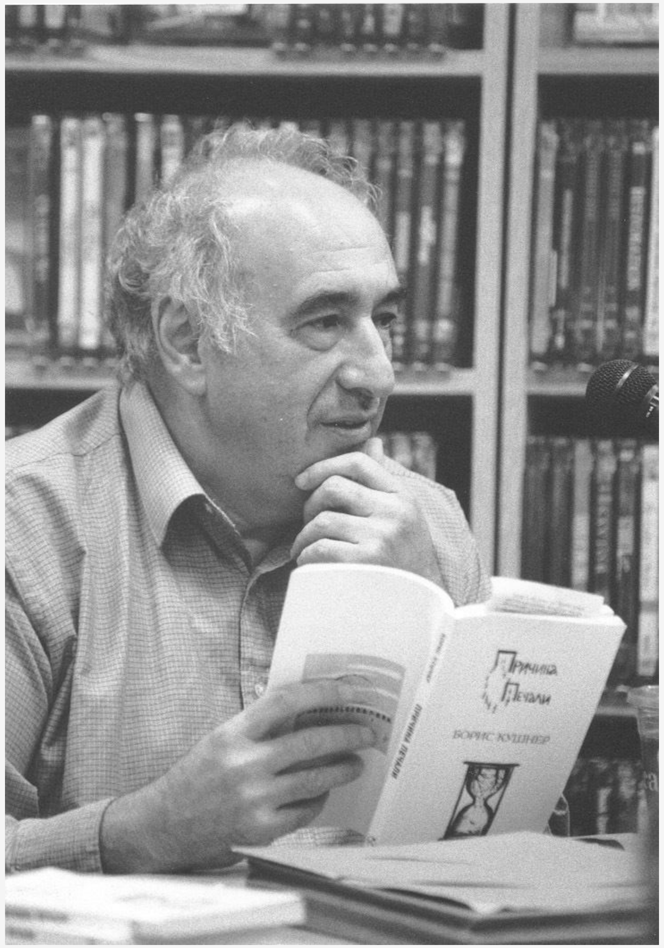
He remembers that when he was ready to enter the department of mathematical logic on the sixteenth floor of the main building of Moscow Lomonosov University, he heard voices inside. Bishop rushed out, followed by Markov with a mystifying smile and one of his closest associates (whom Kushner does not name), who repeated excitedly,

"But he has no standpoint!"

Errett Bishop at his home in La Jolla, California, in February 1983. Photo by Paul Halmos

2. Markov vs. Bishop

Kushner implies that there was a debate between Markov and Bishop in which it was revealed that Markov held a philosophical (methodological) standpoint, whereas Bishop was concerned with "live" mathematical activity without following any elaborate conception. [Kushner 1993, 188-190]



It is natural to assume that Markov's "philosophical (methodological) standpoint," to which Kushner refers, is the "philosophy of constructivism" reconstructed above.

Bur,

Is it true that Bishop lacked a methodological standpoint?

Kushner does not ascribe this view to Markov, but to his anonymous associate.

2. Markov vs. Bishop

At the time of the meeting with Markov in 1966, Bishop was completing his *Foundations of Constructive Analysis*, which was published the following year, in 1967.
In the first chapter of this book [pp. 1-10], entitled "A Constructivist Manifesto," Bishop outlines his position, which Kushner characterized as "an inspired" standpoint [Kushner 1993, 190].

Further, Bishop summarizes the principles of his methodological viewpoint in his work "Schizophrenia in contemporary mathematics" (1985), i.e., 19 years after he met Markov. The title "schizophrenia" refers to the internal division between classicists and constructivists in the foundations and practice of mathematics.

2. Markov vs. Bishop

In this article,

1. Criticizes classical (non-constructive) mathematics

Bishop argues that classical mathematics often asserts the existence of objects without providing a method to construct them. He sees this as a loss of meaning, a detachment from the computational and intuitive content of mathematics.

2. Defends constructive mathematics

Bishop promotes a mathematics where:

- Proofs must yield explicit constructions.
- Existence means "we can build it," and
- Theorems have computational content.

2. Markov vs. Bishop

Admittedly, Bishop's philosophical (methodological) viewpoint was not so elaborate as Markov's. Bishop did not intend to construct an alternative to classical mathematics, nor any alternative logic.

Helen Billinge, in her work "Did Bishop Have a Philosophy of Mathematics?"{2003] gives a precise characterization of Bishop's viewpoint. She states that Bishop

> "did his mathematics in a constructive manner for explicitly philosophical reasons."

However, Bishop's philosophical ideas

> "cannot be rounded out into an adequate philosophy of constructive mathematics" [Billinge 2003, 177].

2. Markov vs. Bishop

1. Logic.

Bishop	Markov	Brouwer
A new, constructive logic, alternative to classical logic, is NOT required.	The construction of constructive logic is a principal issue.	
Does not reject classical logic as a whole.	Classical logic is flawed.	Classical logic is rejected on philosophical grounds.

2. Markov vs. Bishop

2. LEM

Bishop	Markov	Brouwer
Avoids the LEM because it destroys computational meaning, not because it is "wrong."	Accepts a more pragmatic, computational/algorithmic stance, rejecting unrestricted LEM but endorsing principles like Markov's Principle and LEM for decidable predicates. For a *decidable predicate* $P(n)$, LEM holds because one can algorithmically check $P(n)$ or $\neg P(n)$.	The LEM as a general logical principle is rejected on philosophical grounds. Asserting $A \vee \neg A$ without a construction either of A or of $\neg A$ is meaningless. Valid only for decidable/finite cases.

2. Markov vs. Bishop

2. LEM

Bishop	Markov	Brouwer
	For *existential claims* about computable searches, Markov's Principle permits turning a proof of $\neg\neg\exists n\, P(n)$ into an actual n when P is decidable/computable.	

2. Markov vs. Bishop

3. The subject of mathematics

Bishop	**Markov**	**Brouwer**
Mathematics should have numerical meaning and computational content. No metaphysics of the "creating subject." He rejects non-constructive methods on pragmatic grounds.	(Constructive) Mathematics is a technological science (engineering). Mathematics is fundamentally about (normal) algorithms, i.e., mathematics is computable mathematics.	Mathematics is a mental activity grounded in the "creating subject."

2. Markov vs. Bishop

3. The subject of mathematics

Bishop	Markov	Brouwer
Expresses a viewpoint of a specialist primarily interested in the applications of mathematics. However, Bishop builds a human-centered, meaning-driven constructive analysis	Builds a (normal) algorithm-only mathematics. In fact, mathematics is turned into a subfield of computability theory.	

2. Markov vs. Bishop

4. Mathematical objects.

Bishop	**Markov**	**Brouwer**
Real numbers are computable approximations (*effective Cauchy sequences*). In contrast to Markov, Bishop allows non-computable reals as long as they have constructive meaning.	Mathematical objects are outcomes of a process executable in a computer. A real number is a pair of normal algorithms, i.e., real numbers are computable reals only. Non-computable reals do not exist.	Mathematical objects exist only when constructed in the mind. Real numbers are *choice sequences* (potentially infinite, lawless).

2. Markov vs. Bishop

4. Mathematical objects.

Bishop	Markov	Brouwer
Generally, classical structures (sets, functions) are allowed, provided they have constructive meaning.	Rejects classical set theory entirely.	Rejects classical set theory entirely.
Computational.	Computer-generated.	Phenomenological.

2. Markov vs. Bishop

5. Attitude Toward Classical Mathematics

Bishop	Markov	Brouwer
Classical mathematics is valuable but must be reinterpreted constructively. His program is reformist, not revolutionary.	Classical mathematics is fundamentally mistaken. Many classical theorems are meaningless or invalid	
Attempts to rebuild classical mathematics from within.	Declare a program to replace classical mathematics	

2. Markov vs. Bishop

6. Relation to Formal Systems

Bishop	Markov	Brouwer
Feel comfortable with formal systems		Had a negative attitude to formalism and Hilbert-style formalization.

2. Markov vs. Bishop

7. Markov's Principle

Bishop	Markov	Brouwer
Rejects Markov's Principle	Introduced Markov's Principle	Rejects Markov's Principle

Adopting Markov's Principle yields stronger constructive theorems but commits you to an algorithmic ontology.

Rejecting Markov's Principle aligns with Brouwer's purer intuitionism.

2. Markov vs. Bishop

8. Church Thesis

Bishop	Markov	Brouwer
Bishop does not require Church's Thesis to do constructive mathematics; he accepts classical mathematical practice insofar as it can be given constructive reinterpretation.	Reinterprets Church's Thesis in the form of the *Principle of Normalization of Algorithms*	Brouwer frames constructivity in phenomenological terms (*creating subject*, *choice sequences*), so the Church Thesis is not central to his philosophical account. Choice sequences and other intuitionistic principles are not naturally captured by the Church Thesis.

3. Bishop vs. Kushner

It is noteworthy that both Bishop and Kushner authored outstanding monographs on Constructive Analysis. Namely,

- Errett Bishop, *Foundations of Constructive Analysis* (1967).
- Boris A. Kushner *Lectures on Constructive Mathematical Analysis*. Moscow 1973. Translated in 1984 by the American Mathematical Society.

Kushner presents an algorithmic, constructive approach in the Markov style that is complementary but philosophically and methodologically distinct from Errett Bishop's *Foundations of Constructive Analysis*.

Bishop develops a pragmatic, numerically-oriented constructivist program.

4. Early foundational discussions in Moscow

The foundational debates in Moscow did not revolve around the paradoxes of set theory.

After 1918, the work of Gottlob Frege (1848–1925) and the functional approach to logic remained unknown for a long time. Frege’s works were systematically studied for the first time in 1959 by Boris V. Birjukov (1922–2014), at the suggestion of Sofia A. Yanovskaja (1896–1966).

During the early Soviet period, there was no immediate experience of the sensational situation that emerged in the West with the discovery of logical paradoxes in Frege’s system and the subsequent formulation of alternative foundational programmes for the reconstruction of mathematics.

4. Early foundational discussions in Moscow

Thus, foundational studies in the Soviet Union/Russia

- were NOT focused on the elimination of paradoxes from set theory.

Instead,

- They were focused on the rejection of the actual infinite from mathematics and the admission of a working version of the potential infinite.
  In particular,
  There was much controversy about the use of the actual infinite, including the actual infinite of the series of natural numbers, and the use of the axiom of choice in the construction of the continuum (Nikolai Nikolayevich Luzin (1883–1950)).

Luzin's views were formed mainly under the influence of the French function-theorists, notably Émile Borel (1871–1956). For this reason, Troelstra and van Dalen included him in the group of French semi-intuitionists.

4. Early foundational discussions in Moscow

Concerning the admissibility/rejection of the actual infinite, the following viewpoints were developed in Moscow:

i. Pavel A. Florensky (1882–1937), for whom the actual infinite is the key to the problem of celestial hierarchy [theological philosophy].
ii. N.N. Luzin, for whom the unlimited use of the actual infinite and the axiom of choice in mathematics could lead to conclusions devoid of epistemological meaning [quasi-intuitionism].

Luzin adopted a decisive view against the actual infinite, including the infinity of the set of natural numbers. In a letter to Pavel A. Florensky, dated August 1915, Luzin writes

> "There is no actual infinite! When we dare to talk about it, we, in fact, always talk about the finite and about the fact that after *n* there is *n*+1 … that's all!" [Demidov 2018a, 342].

4. Early foundational discussions in Moscow

The same idea is repeated in his *Leçons sur les ensembles analytiques et leurs applications*

> "What we call the actual infinite is (or would be) only the fixed and very large finite." [Lusin 1930, 322]

To Luzin is also ascribed the following bold statement

> "The series of natural numbers does not seem to be an absolutely objective structure. It seems to be an artifact of the brain of the mathematician who happens to be speaking about the natural numbers" [Demidov, S. S.; Levshin 2016 (1999), 12]

What is meant is the actual infinite of the series of natural numbers, which is viewed by Luzin as a creation of the mathematician's mind.

5. Luzin's tradition

Luzin's work on analytic sets led him directly into foundational questions about the continuum and the actual infinite.

In his 1927 monograph *Sur les ensembles analytiques*, Luzin introduced the axiom of partition (*l'Axiome du Partage*), asserting that

Every "tractable" equivalence relation on the continuum admits a transversal (a choice of one representative from each equivalence class).

Luzin remarked that the justification for this axiom is no better than the justification for the Axiom of Choice itself.

This is possibly the earliest documented moment in the Moscow school where the Axiom of Choice explicitly causes caution.

5. Luzin's tradition

Luzin's descriptive set theory was deeply tied to the structure of the continuum. So, his hesitation about the Axiom of Choice reflects a broader anxiety:

- The Axiom of Choice allows the construction of non-measurable sets, which threatened the regularity properties central to Luzin's program.

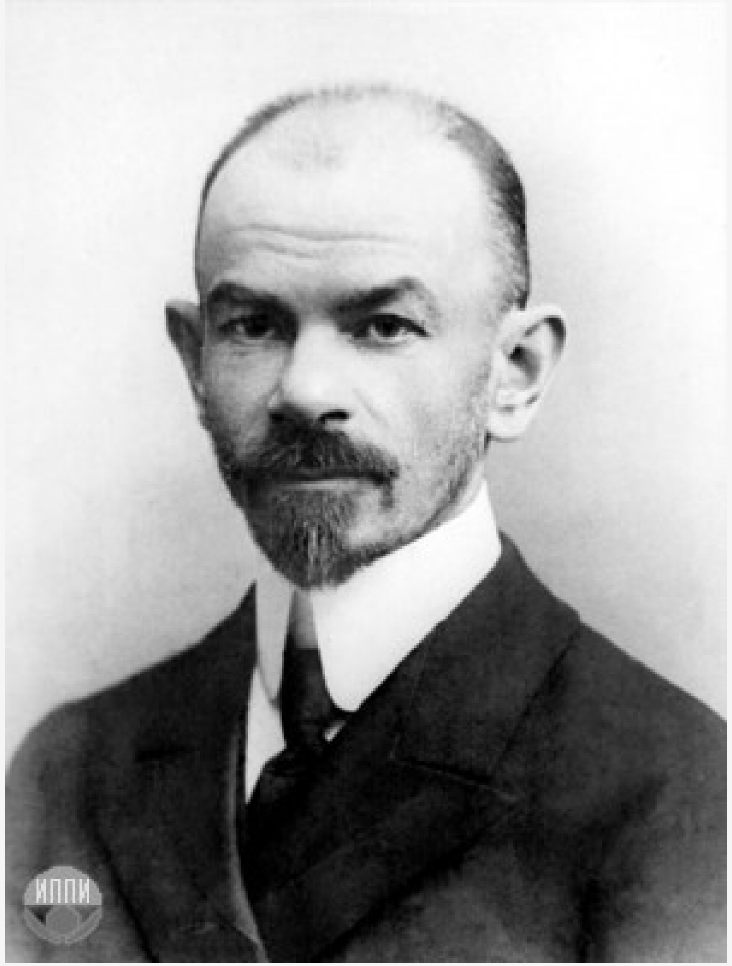
Thus, the debate over the Axiom of Choice in Moscow began not as a purely logical issue, but as a conflict between set-theoretic freedom and analytic regularity. Hence, Luzin's school tradition emphasized limits on choice-style principles because they produce pathological counterexamples.

5. Luzin's tradition

What was Luzin's attitude towards intuitionism?

Luzin was sympathetic, deeply engaged, but ultimately ambivalent towards intuitionism. His attitude was a mixture of attraction, critique, and personal struggle.

Luzin was profoundly influenced by Dmitri Fyodorovich Egorov (1869–1931), whose religious and philosophical commitments made him receptive to non-classical foundations.



5. Luzin’s tradition

Luzin inherited this atmosphere.

He was drawn to the adoption of

- the primacy of mental constructions,
- the suspicion toward non-constructive existence proofs,
- the idea that mathematics must be grounded in inner evidence (очевидность).

This is visible in his writings on the concept of function, where he emphasizes the *psychological* and *intuitive* origins of mathematical notions.

5. Luzin's tradition

For Luzin, intuition was not merely psychological—it was spiritual, shaped by Pavel A. Florensky, who emphasized

- inner illumination,
- the direct experience of truth, and
- the unity of spiritual and intellectual intuition.

Hence, mathematical insight required an inner clarity that was associated with his religious worldview. Intuition was not just a cognitive act but a mode of spiritual viewing.

Intuition was the source of mathematics, the place where mathematical objects are "viewed" (by the "mind's eye") or "brought forth."

5. Luzin's tradition

Luzin admired Brouwer's philosophical courage and saw intuitionism as a serious attempt to reconnect mathematics with its experiential roots.

Nevertheless, Luzin never became an intuitionist. This is possibly due to various reasons:

- He was too committed to classical analysis.
  His life's work (descriptive set theory, measurable functions, analytic sets) relies heavily on classical logic and non-constructive methods.
- He resisted Brouwer's rejection of classical logic.
  He found the intuitionistic ban on the law of excluded middle too restrictive for the kind of mathematics he wanted to build.
- He preferred a pluralistic view.
  Rather than choosing sides, he believed different foundational approaches illuminate mathematics in different ways.

5. Luzin's tradition

Luzin's writings show a distinctive, nuanced stance:

- He considers intuition to be the origin of mathematics, but formal methods are the engine of mathematical progress.
- He saw intuitionism as a *reminder* of mathematics' psychological roots, but not as a *replacement* for classical reasoning.

In this sense, Luzin's position is closer to Poincaré than to Brouwer:

- Intuition is indispensable, but classical logic remains valid within its proper domain.

6. Rashevskij on the natural number series

By referring directly to Luzin, Petr Konstanovich Rashevskii (1907–1983) states some original reflections on the nature of the series of natural numbers.

In his view, the mathematical idea of the actual infinite of natural numbers is not needed in natural science. Physicists need a mathematical theory of natural numbers, in which the numbers could acquire a "blurred meaning" when they become very great. For a mathematician, the addition of a unit changes the number, but for a physicist, what changes if a molecule were added to a container with gas? If we adopt this idea, then we have to abandon our standard concept that any number of the natural series can be obtained by successive counting of units.

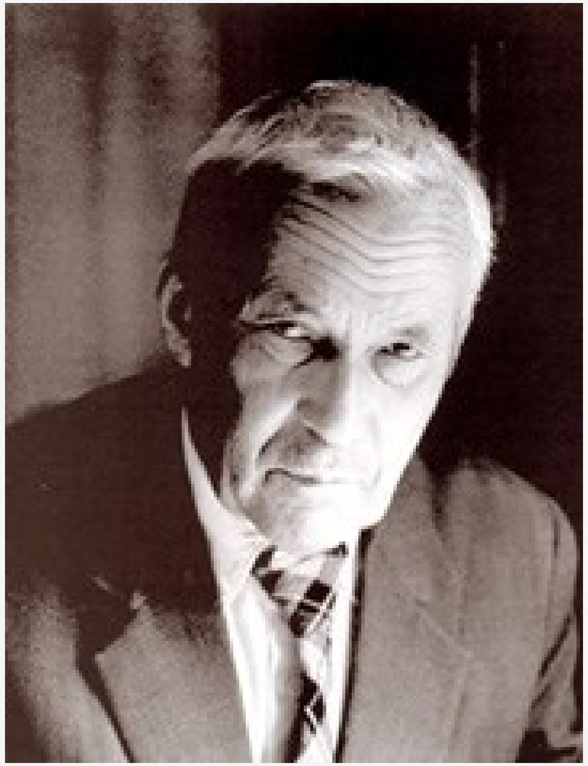
6. Rashevskij on the natural number series

Rashevskij notes that this conception challenges the principle of mathematical induction, and the "numbers" of such a hypothetical natural series would be objects of another nature [Rashevskij 1973, 244].

These reflections anticipate, in some sense, Esenin-Vol'pin's ideas on the natural series and his criticism of the principle of mathematical induction, as well as ideas advanced within the strict finitist approach concerning the Sorites paradox and vague predicates [Magidor 2012, Dean 2018].

7. Kolmogorov's problem-oriented semantics

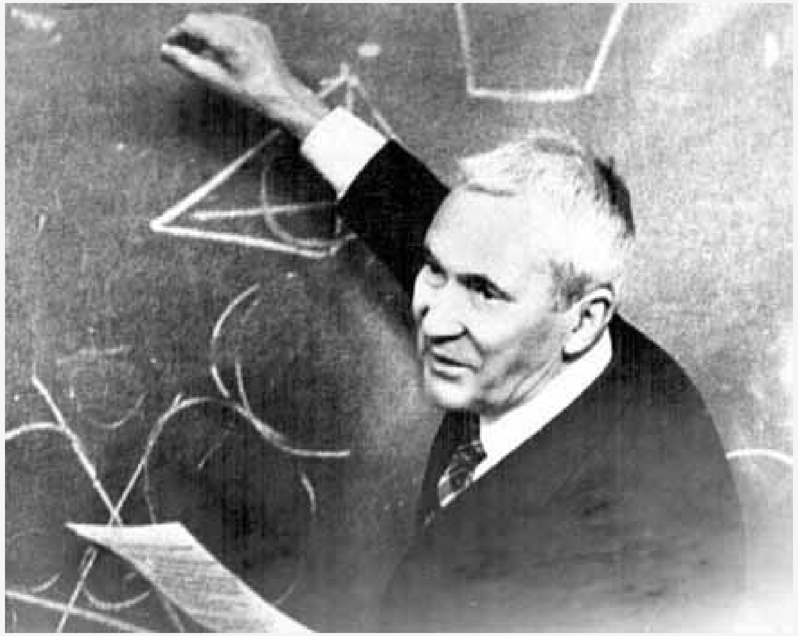
A substantial shift in the foundational debates was attained by Andrei Nikolaevich Kolmogorov (1903-1987), with his early works on intuitionistic logic



- 1925. «O principe *tertium non datur*». *Matematicheskij sbornik* 32 (4), 646-667. English translation "On the principle *tertium non datur*", in van Heijennort, Jean (Ed.) 1967. *From Frege to Gödel, A Source Book in Mathematical Logic, 1879-1931*.
- 1932. "Zur Deutung der intuitionistischen Logik", *Math. Zeitschrift* 35, 58-65.

7. Kolmogorov's problem-oriented semantics

Kolmogorov was one of Luzin's most brilliant students, and their teacher–student relationship shaped Soviet analysis, probability, and foundational debates, notably the controversies around the continuum, the actual infinite, and the Axiom of Choice.



Luzin remained primarily a function-theorist and set-theorist concerned with regularity properties of sets and the technical consequences of accepting strong choice principles.

However, Kolmogorov introduced the problem of the LEM in the Moscow foundational debates.

7. Kolmogorov's problem-oriented semantics

In his paper, "On the principle *tertium non datur*" (1925), he argues that the LEM is not acceptable when one requires constructive methods.

Kolmogorov did not simply endorse or reject the LEM outright but reinterpreted its role by developing a problem-oriented semantics (the Calculus of Problems) that explains why the LEM fails for constructive problem-solving while remaining acceptable in classical contexts.

P.S. Aleksandrov and A.N. Kolmogorov. Donated by I.G. Bashmakova.

7. Kolmogorov's problem-oriented semantics

Kolmogorov's Calculus of Problems (1932) offers a semantic reading in which propositions are *problems* (*задача* – task, Aufgabe) to be solved (*fulfilled*) and logical connectives are operations on problems. Kolmogorov does not provide a definition of what a problem (*task*) is.

Logical connectives are interpreted as operations on problems:

- $A \wedge B$: solve both problems.
- $A \vee B$: solve at least one problem.
- $A \rightarrow B$: a method transforming any solution of *A* into a solution of *B*.
- $\neg A$: a method turning any solution of *A* into a contradiction.

In contrast to Markov's connectives, which are computational, Kolmogorov's are problem-theoretic.

7. Kolmogorov's problem-oriented semantics

Under this view, negation and hence the LEM acquire an operational meaning:

- $\neg A$ is a method that transforms any putative solution of A into a contradiction, and
- $A \vee \neg A$ is a demand for a method that either solves A or converts any attempted solution into a refutation.

Because such a uniform method need not exist, the LEM is not generally validated in the calculus of problems. This explains the intuitionistic rejection of the LEM from a problem-theoretic perspective.

7. Kolmogorov's problem-oriented semantics

Kolmogorov died in 1987, leaving behind a huge legacy in many mathematical areas, including logic.

7. Kolmogorov's problem-oriented semantics

Kolmogorov's interpretation of intuitionistic logic is considered by some logicians to be one of the historical sources of the Brouwer–Heyting–Kolmogorov (BHK) reading of intuitionistic logic:

- proofs as *constructions*,
- propositions as *tasks*, which are required to be *fulfilled* by *carrying out constructions*, and
- connectives as constructions on tasks.

Strictly speaking, Kolmogorov's interpretation is not philosophically the same as Heyting's. In Heyting's view, propositions are *intentions* (towards constructions) that are required to be *fulfilled* (made true) or *disappointed* (made false) by our mathematical constructions.

If we assume there is no essential difference between the two readings (van Atten), then we obtain the BHK interpretation. Nevertheless, one difference remains: Kolmogorov's approach is problem-based, whereas Heyting's is axiomatic.

7. Kolmogorov's problem-oriented semantics

Thus, Kolmogorov reframed the debate by showing that whether the LEM holds depends on

- what one takes propositions to be (truth-bearers or problems), and
- what one requires of a proof (existence vs. explicit method).

Unlike a simple repudiation of the LEM, Kolmogorov's approach explains why the LEM is meaningful classically but fails constructively, because the constructive notion of "method" is stricter than the classical notion of truth by truth-value alone.

This shift from *truth* to *solvability* remains central to modern constructive and proof-theoretic perspectives.

Kolmogorov's paper also marks a shift from the axiomatic to the problem-based model of organization of a theory. [Drago Antonino 2022]

8. Kolmogorov and Luzin

Was Kolmogorov's interpretation of intuitionism a break from the Luzin tradition?

Kolmogorov's "problem-semantics" is an intellectual descendant of Luzin's emphasis on *mental acts* and constructive *insight,* but stripped of its religious and mystical overtones inherited by Luzin (due to Florensky).

Instead of Luzin's grounding of intuition in spirituality, Kolmogorov grounded it in operational clarity. In this sense, we can claim that Kolmogorov essentially formalized Luzin's phenomenology. Thus, Luzin's spiritual-phenomenological conception of intuition becomes, in Kolmogorov's hands, a rigorous logical calculus of problems. Kolmogorov's intuitionism is thus not a break with Luzin, but one of Luzin's most profound intellectual legacies.

9. Kolmogorov vs. Florensky and Luzin

Florensky-Luzin's influence is traceable in Kolmogorov's treatment of intuitionism

Kolmogorov's 1925 paper "On the Principle of the Excluded Middle" breaks from Florensky-Luzin's mystical metaphysics, but its phenomenology of mathematical acts and its reinterpretation of logic as a theory of experience echo Florensky's epistemology, declared especially in *The Pillar and Ground of the Truth* (1914) and his later essays.

10. Kolmogorov vs. Florensky: a semiotic perspective

Specifically, the following theses held by Florensky can be interpreted as anticipating Kolmogorov's semantics:

- Truth as an act of realization (fulfillment). [*Pillar and Ground*, Letter II ("Truth")]
  - Florensky argues that truth is not a static proposition but an act of fulfillment. He insists that truth is performed, knowledge is participatory, and meaning arises only when the knower realizes the content.
  - This anticipates Kolmogorov's position that a proposition is not a truth-value but a problem whose meaning is the method that solves it.

10. Kolmogorov vs. Florensky: a semiotic perspective

- The critique of abstract dichotomies. [*Pillar and Ground*, Letter IV ("Doubt")]
  - Florensky attacks the idea that reality is exhausted by binary oppositions. He argues that many dichotomies are false abstractions that outrun lived experience.
  - Florensky's critique of premature dichotomy becomes Kolmogorov's critique of non-constructive disjunction and thereby LEM.

10. Kolmogorov vs. Florensky: a semiotic perspective

- Meaning arises from the act that fulfills the symbol. [*Pillar and Ground*, Letter VII ("Symbol")]
  - Florensky's theory of symbol is deeply operational:
    - a symbol has meaning only when it is activated,
    - its content is the experience that realizes it,
    - symbols are demands for acts, not inert signs.
  - This is astonishingly close to Kolmogorov's interpretation of logical connectives:
    - Conjunction is the simultaneous solvability of two tasks.
    - Disjunction is the solvability of at least one task.
    - Implication is a method transforming solutions of one task into solutions of another.
    - Negation is the impossibility of a method.

10. Kolmogorov vs. Florensky: a semiotic perspective

- Truth as "energetic" rather than propositional. [*The Doctrine of Truth* (1914–1917 essays)]
  - Florensky distinguishes between:
    - *energetic truth* (truth as lived act),
    - *logical truth* (truth as propositional form).

    He argues that logical truth is derivative and secondary.
  - Kolmogorov's semantics is built on a parallel hierarchy:
    - The act (method) is primary.
    - The proposition is secondary and defined through the act.

10. Kolmogorov vs. Florensky: a semiotic perspective

- Thought as a sequence of operations, not static states. [*On the Organism of Thought* (unpublished notes, 1910s)]
  - Florensky describes thinking as:
    - a chain of operations, each step of which is transforming the previous one,
    - The meaning emerges from the process, not the result.
  - Kolmogorov's interpretation of implication as a method transforming solutions is a direct formal analogue.

10. Kolmogorov vs. Florensky: a semiotic perspective

- Antinomies arise when logic outruns experience. [*The Antinomy of Truth* (1914)]
    - Florensky argues that antinomies appear when logical form is detached from experiential content.
    - Kolmogorov's intuitionistic stance can be interpreted as close to Florensky's thesis:
        - classical logic outruns constructive experience.
        - The LEM is valid only when a method exists.

To sum-up
For Florensky, truth is the act that fulfills meaning.
For Kolmogorov, a proposition is a problem whose solution gives its meaning and establishes its truth.

10. Kolmogorov vs. Florensky: a semiotic perspective

In Kolmogorov's Calculus of Problems,

- Propositions are interpreted as problems, which are not truth-value bearers, but tasks demanding an act (realization). There is no assertion without realization (fulfillment of the task).
  For Florensky, truth is an act of realization. There is no truth without realization.
- A proof is not a derivation but a method to fulfill the demand.
  For Florensky, knowledge is participatory, and for Kolmogorov, proof is the method of participating in a problem (fulfilling the task).
- In Kolmogorov, meaning arises from the experiential act that solves the problem. In Florensky, meaning arises from lived experience.

Thus, Kolmogorov's logic of the Calculus of Problems is a logic of experiential acts, not of abstract truth-values.

10. Kolmogorov vs. Florensky: a semiotic perspective

Moreover,

- Florensky's entire philosophical method is built on the principle:
  The act precedes the concept. Life precedes logic.
- Kolmogorov's 1925 paper embodies this principle mathematically:
  - A proposition has meaning only through the *act* that solves it.
  - Logical connectives are defined through *operations*, not truth tables.
  - Negation is the impossibility of a method, not a truth-value.
  - Implication is a *transformation of acts*, not a relation between propositions.

10. Kolmogorov vs. Florensky: a semiotic perspective

Hence, Florensky's unique phenomenology of knowledge was translated into Kolmogorov's constructive semantics. Where Florensky spoke of spiritual acts, and Luzin of intuitive acts, Kolmogorov spoke of problem-solving acts. The structure is the same; the content is transformed.

This is why Kolmogorov's interpretation of intuitionism is so different from Brouwer's.

It is not Dutch mysticism. It is not German formalism.

It is Moscow (theological) phenomenology, refracted through Luzin and ultimately rooted in Florensky.

11. Kolmogorov vs. Markov

Markov's *Constructive Mathematics* and Kolmogorov's *Calculus of Problems* represent two traditions that are fundamentally different in motivation, ontology, and logical interpretation.

- Markov's Constructive Mathematics is computable mathematics; only algorithmically generable objects exist; proofs correspond to effective procedures (normal algorithms).
- Computability is central.

- Kolmogorov's Calculus of Problems is a logical theory of problems;
- propositions are interpreted as problems (*tasks*), and logical connectives describe operations on problems.
- Computability is not essential.
  A "method" may be non-algorithmic, not necessarily recursive.

These are not two versions of the same idea; they belong to different conceptual universes.

11. Kolmogorov vs. Markov

- The ontology of Markov's Constructive Mathematics consists of computable objects. It does not include non-computable objects, e.g., non-computable reals, or, generally, objects whose existence is provable non-constructively.
- In Markov's Constructive Mathematics, proofs correspond to effective procedures (normal algorithms).

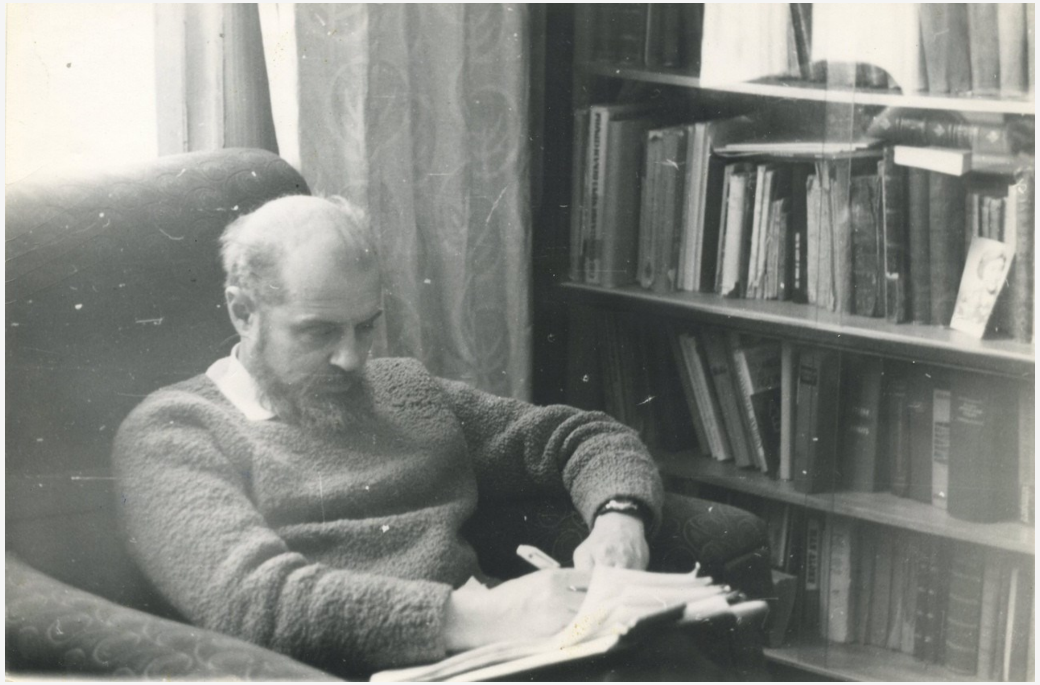
- The ontology of Kolmogorov's Calculus of Problems consists of problems (*tasks*), not objects.
- In Kolmogorov's Calculus of Problems, a proof is a method for solving the problem. No commitment to computability; the notion of "method" is pre-formal and broader than an algorithm.

Conclusion: Markov builds a fully algorithmic mathematics; Kolmogorov builds a semantics of problems that explains intuitionistic logic without restricting mathematics to computable objects.

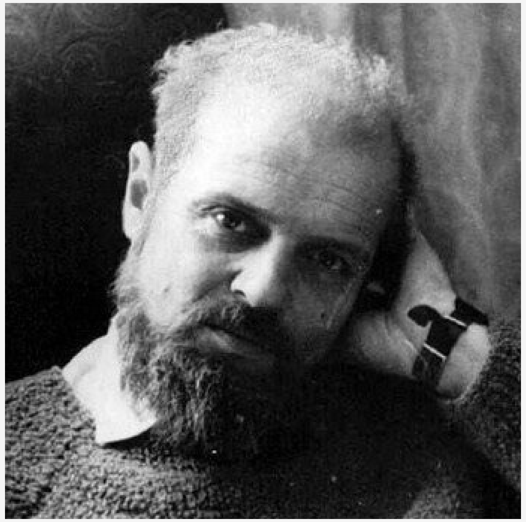
12. Markov vs. Aleksandr S. Esenin-Vol′pin

A more radical form of finitism was proclaimed by **Aleksandr S. Esenin-Vol'pin** (1924–2016).

- Esenin-Vol'pin rejected Markov's abstraction of potential infinity and the related principle of potential realizability.
- He suggested substituting Markov's principle with his principle of factual (practical) realizability, requiring that mathematical existence claims be grounded on methods that are actually performable in practice, i.e., finitary, feasible constructions rather than merely "in principle" or "asymptotically realizable" procedures. The construction or verification must be carried out within concrete, physically or cognitively feasible bounds.



12. Markov vs. Aleksandr S. Esenin-Vol′pin



Esenin-Vol'pin's work was neglected in the Soviet Union, although it had attracted the attention of Kurt Gödel and Paul Bernays.

Only Sofya Aleksandrovna Yanovskaya (1896–1966) wrote extensive reviews on it in 1959.

- She presented Esenin-Vol'pin's results on axiomatic set theory [Esenin-Vol'pin 1954, 1957], notably, his version of axiomatic set theory without the axiom of choice, in which the continuum hypothesis and Suslin hypothesis are not derivable.
- She criticized his concept of *factual realizability*, noting that it is not evident from the classical point of view and cannot be formalized within a traditional logical system.
- She also outlined Esenin-Vol'pin's foundational program for the proof of the consistency of classical set theory by ultra-finitistic means. A reconstruction of Esenin-Vol'pin's consistency proof was given by Gandy [1982].

12. Markov vs. Aleksandr S. Esenin-Vol′pin

Esenin-Vol'pin's rejection of Markov's principle of potential realizability was also criticized by Vadim A. Yankov (1935–2024), a constructivist of Markov's school.

He noted that

- If Markov's principle is rejected, there would exist many "sequences of natural numbers" that are not isomorphic. One of them might be "shorter" or "lengthier" than the other. Although this situation might seem paradoxical, there is no contradiction in it.

Thus, Yankov concludes that this kind of mathematics is legitimate.

12. Markov vs. Aleksandr S. Esenin-Vol'pin

Esenin-Vol'pin interprets the classical Zermelo-Frenkel set-theoretic mathematics in ultra-intuitionistic terms on the assumption of the existence of different sequences of natural numbers so that some of them are closed under certain operations and cannot reach other sequences. In this way, classical mathematics is proved consistent from the standpoint of ultra-intuitionism.

However, Yankov notes that appealing to assumptions diminishes the value of the proof. To support his view, Yankov also communicates Pyotr Novikov's similar personal opinion. In conclusion, Yankov considers Esenin-Vol'pin's ultra-intuitionistic program an unfinished project, possibly not unified but combining a variety of approaches.

13. Saint-Petersburg vs. Moscow School

The Moscow school (Egorov, Luzin, Urysohn, Alexandrov, Kolmogorov) was unusually receptive to foundational debates. Luzin's students explored:

- constructive mathematics (Novikov),
- intuitionistic logic (Kolmogorov's 1925 paper),
- descriptive set theory (Luzin himself).

Luzin encouraged this diversity. He did not impose a foundational doctrine.

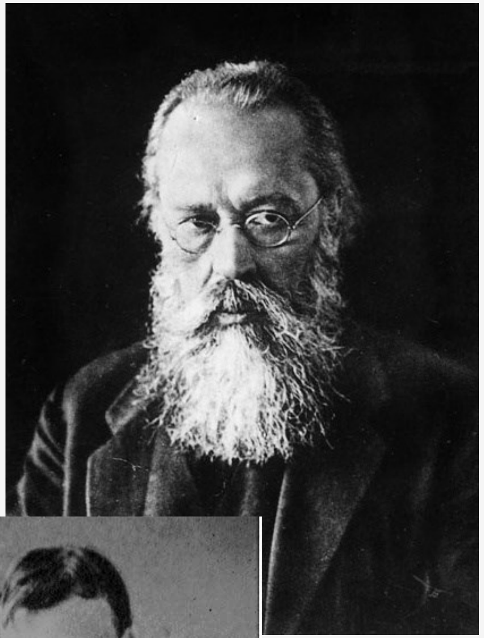
The mathematicians of the Moscow school were tolerant of various philosophical conceptions of mathematics and mathematical objects, in contrast to the mathematicians of the Saint-Petersburg school, whose values were primarily positivist and Western-oriented. Thus, Markov's reaction to intuitionism goes beyond the horizon defined by Kolmogorov's interpretation of intuitionistic logic.

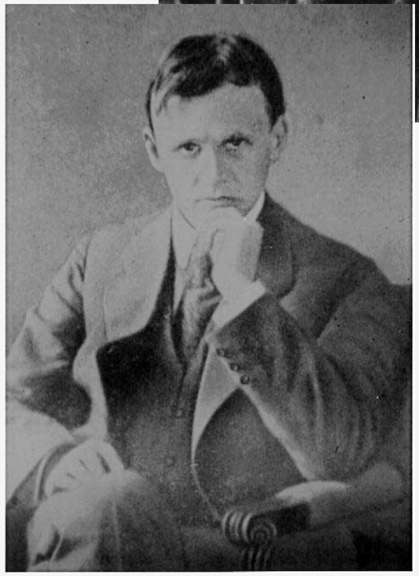
13. Saint-Petersburg vs. Moscow School

This discrepancy between the two schools assumed the form of open conflicts throughout history, especially after Pafnuty Lvovich Chebyshev's (1821–1894) death (e.g., in the 1890s).

A vivid expression of the opposition between the two schools is the alleged judgment of the Professor of Saint Petersburg Academy, Vladimir Andreevich Steklov (1864–1926), on young N.N. Luzin's dissertation *Integral and trigonometric series*. After browsing the work of his Moscow colleague, he asked:

"Where are the formulas here? This is not mathematics, but some philosophy!"





13. Saint-Petersburg vs. Moscow School

Markov emerged from the intellectual environment of the Saint Petersburg –Leningrad school. All his educational background and early career are connected with Leningrad.

In 1935, he was awarded a Doctor of Science degree at Leningrad University; the next year, he was nominated Professor at the same university. Since 1939, he worked at the Leningrad Branch of the Steklov Mathematical Institute.

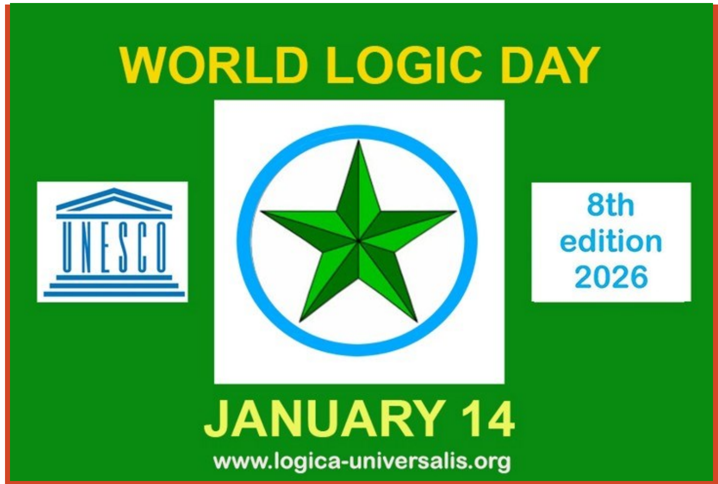
His research at Leningrad was initially focused on the general theory of dynamical systems and related problems in topology and measure theory, particularly algorithmic problems in topology, theory of computable invariants of binary relations, cryptography, etc. Since 1946, he has turned to the theory of algorithms and recursive functions, leading to the introduction of the concept of a *normal algorithm*.

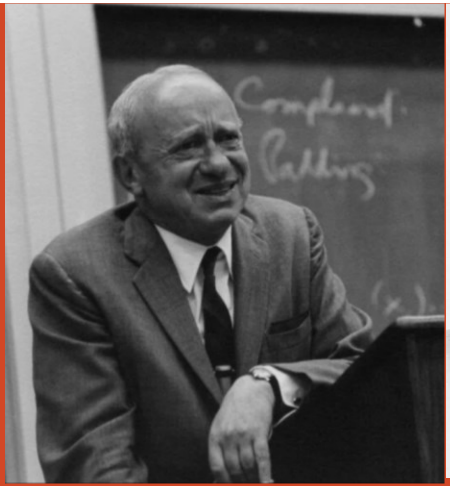
13. Saint-Petersburg vs. Moscow School

Moving to Moscow in 1955, Markov brought with him the arsenal of his theory of (normal) algorithms as well as the traditional positivist dispositions of the Leningrad school's intellectual environment.

By his "algorithmic" approach, he attempts to "free" intuitionism from its underpinning metaphysical assumptions. Markov's attitude can be better explained as the impact of a positivist disposition and his earlier research on algorithmic problems during the period of his life spent in Leningrad (1933-1955).

Thus, Markov's constructive mathematics was the outcome of his research orientation on algorithmic problems and the influence of the mathematical style and values prevailing in the Saint Petersburg mathematical school, which is characterized by the proclaimed primacy of applications and the search for rigor and effective solutions.





From Brouwer’s intuitionism to Kolmogorov’s problem-oriented semantics, Markov’s algorithmic vision and Bishop’s pragmatic program, constructivists disagreed passionately. Understanding their differences enables us to appreciate the richness—not the fragmentation—of constructivist thought.

THE DEBATE CONTINUES